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On a conjecture of Ilmonen, Haukkanen and Merikoski concerning the smallest eigenvalues of certain GCD related matrices

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ABSTRACT

Let K_n be the set of all $n \times n$ lower triangular $(0,1)$-matrices with each diagonal element equal to 1, $L_n = \{YY^T : Y \in K_n\}$ and let c_n be the minimum of the smallest eigenvalue of YY^T as Y goes through K_n. The Ilmonen–Haukkanen–Merikoski conjecture (the IHM conjecture) states that c_n is equal to the smallest eigenvalue of $Y_0 Y_0^T$, where $Y_0 \in K_n$ with $(Y_0)_{ij} = \frac{1-(-1)^{i+j}}{2}$ for $i > j$. In this paper, we present a proof of this conjecture. In our proof we use an inequality for spectral radii of nonnegative matrices.

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1. Introduction

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers, (x_i, x_j) denote the greatest common divisor of x_i and x_j and let ε be a positive real number. The $n \times n$ matrices $(S) = ((x_i, x_j))$ and $(S^\varepsilon) = ((x_i, x_j)^\varepsilon)$ are called the GCD matrix and the power GCD matrix on S, respectively. The LCM matrix and the power LCM matrix are similarly defined. In 1876, Smith [30] proved that if S is factor closed, then $\det(S) = \prod_{k=1}^{n} \varphi(x_k)$, where φ is Euler's totient. Since then many results on these matrices have been published in the literature, see e.g. [5,6,10,11,13,21].

An interesting and active area in the study of GCD type matrices is their eigenstructure. The first results on this subject were published in the papers [32,7,23] but the paper of Hong and Loewy [16] can be considered as the first paper on the study of the eigenvalues of GCD and related matrices due to the number theoretical aspect of the subject. Since their pioneering paper many results on the subject have been published in the literature, see e.g. [1,3,4,14,15,17,19,24,26–28]. In that paper, Hong and Loewy investigated the asymptotic behavior of the eigenvalues of power GCD matrices by using some tools of number theory. Beside their results on asymptotic behavior of these matrices, in the same paper Hong and Loewy introduced a constant c_n and used it to present a lower bound for the smallest eigenvalues of power GCD matrices. Let K_n be the set of all $n \times n$ lower triangular $(0,1)$-matrices with each diagonal element equal to 1 and let $L_n = \{YY^T : Y \in K_n\}$. They defined the numbers c_n depending only on n as follows:

$$c_n = \min_{Z \in L_n} \left\{ \mu_n^{(1)}(Z) : \mu_n^{(1)}(Z) \text{ is the smallest eigenvalue of } Z \right\}. \tag{1.1}$$

Then they proved that

$$\lambda_n^{(1)}((S^\varepsilon)) \geq c_n \cdot \min_{1 \leq i \leq n} \{J_\varepsilon(x_i)\},$$

where J_ε is Jordan's generalization of Euler's totient and $\lambda_n^{(1)}((S^\varepsilon))$ is the smallest eigenvalue of the power GCD matrix (S^ε), see [16, Theorem 4.2].

In 2008, Ilmonen, Haukkanen and Merikoski [19] studied eigenvalues of meet and join matrices which are abstract generalizations of GCD and LCM matrices, respectively, and they generalized the above result concerning the numbers c_n for positive definite meet matrices defined on locally finite meet semilattices. They also obtained a similar result for join matrices and hence, in particular, for LCM matrices. In the same paper, in the light of their MATLAB calculations for $n = 2, 3, \ldots, 7$, they presented an interesting conjecture about the constants c_n.

Conjecture 1.1 *(The IHM conjecture). (See [19, Conjecture 7.1].) Let $Y_0 = (y_{ij}^0) \in K_n$ be defined by*

$$y^0_{ij} = \frac{1-(-1)^{i+j}}{2} \tag{1.2}$$

if $i > j$. Then c_n is equal to the smallest eigenvalue of $Y_0 Y_0^T$.

We have further numerical evidence of the IHM conjecture as follows. Recently, the first, the second and the fourth author of the paper have investigated that the IHM conjecture holds for $n = 7$ and $n = 8$ with the help of a MATLAB code. Our MATLAB code running on a computer[1] has verified the truth of the IHM conjecture for $n = 7$ in 23 minutes and for $n = 8$ in 3.5 days. Since $|L_8| = 2^{28} = 268{,}435{,}456$ and $|L_9| = 2^{36} = 68{,}719{,}476{,}736$, it would take about 3 years to verify the IHM conjecture for $n = 9$ with the help of our MATLAB code. To overcome this difficulty about time, we write a different code in C programming language. We use Newton's identities (see [20]) to obtain the characteristic polynomial of a matrix Z in L_n and we calculate the smallest eigenvalue of Z by using Newton's method (see [31]) to shorten the running time. Indeed, our C code running on the same computer has verified the truth of the IHM conjecture in 30 minutes for $n = 8$ and in 7 days for $n = 9$. Thus, we have concluded that Conjecture 1.1 holds for $n = 8$ and $n = 9$. This investigation has been presented by the first author of this paper in [2].

After obtaining enough numerical evidence that the IHM conjecture can be true, we get the motivation to find out a proof of it. The strategy of the proof is as follows. We prove that for any matrix Y in K_n, $|Y^{-1}| \le |Y_0^{-1}|$, where the matrix Y_0 given by (1.2) and $|Y^{-1}|$ is the element-wise absolute value of the matrix Y^{-1}. Secondly, we show that $|Z^{-1}| \le |(Y_0 Y_0^T)^{-1}|$ for all $Z \in L_n$. Then, by using an inequality for spectral radii of nonnegative matrices, we obtain a proof of the IHM conjecture. We conclude the paper with a conjecture on the uniqueness of the matrix Y_0 and a discussion about further studies on the constant c_n.

2. Properties of matrices in K_n

First we present a simple fact about a particular nilpotent $(0,1)$-matrix which we use in the course of our proofs. Here we give the proof of this fact though one can find in the literature.

Lemma 2.1. *Let $Y \in K_n$ and $N := Y - I$, where I is the $n \times n$ identity matrix. We denote by $(N^k)_{ij}$ the ij-entry of the positive integer k-th power of N. Then we have the following properties.*

- **i)** $(N^k)_{ij} = 0$ *whenever* $i - j < k$,
- **ii)** $Y^{-1} = I - N + N^2 - \cdots + (-1)^{n-1} N^{n-1}$.

[1] Intel Core i7-920 Quad Core 2.66 GHz 8 MB L Cache 24 GB DDR3 RAM.

Proof. Since N is a strictly lower triangular $(0, 1)$-matrix, the proof of the first claim follows from the matrix multiplication. For the proof of the second claim, consider

$$I^n - (-N)^n = (I + N)(I - N + N^2 - \cdots + (-1)^{n-1}N^{n-1}).$$

Since $N^n = 0$ and $Y = I + N$, we have

$$Y^{-1} = I - N + N^2 - \cdots + (-1)^{n-1}N^{n-1}. \qquad \square$$

Now we investigate the inverse of any matrix Y in K_n. In the following lemma we obtain a recurrence relation for the entries of the inverse of Y.

Lemma 2.2. *Let $Y \in K_n$ and $N := Y - I$. Denote $N = (n_{ij})$ and $Y^{-1} = (a_{ij})$. Then we have the following recurrence relation for a_{kl}:*

$$a_{kl} = \begin{cases} 0 & \text{if } k < l, \\ 1 & \text{if } k = l, \\ -\sum_{i=l}^{k-1} n_{ki}a_{il} & \text{if } k > l. \end{cases}$$

Proof. When we multiply both sides of the equality in Lemma 2.1(ii) from the left by $-N$, we have

$$-NY^{-1} = -N + N^2 - \cdots + (-1)^{n-1}N^{n-1} + (-1)^n N^n.$$

Since $N^n = 0$, one can easily obtain

$$I - NY^{-1} = Y^{-1}. \tag{2.1}$$

By Lemma 2.1, it is clear that $a_{kl} = 0$ if $k < l$ and $a_{kl} = 1$ if $k = l$. Now, from (2.1), we have

$$a_{kl} = -\sum_{i=1}^{n} n_{ki}a_{il}$$

for all $k > l$ and hence, by Lemma 2.1, we obtain

$$a_{kl} = -\sum_{i=l}^{k-1} n_{ki}a_{il}$$

for all $k > l$. $\square$

In the following theorem, we find an upper bound for each a_{ij} in terms of Fibonacci numbers which is a surprising result.

Theorem 2.1. *Let $Y \in K_n$ and denote $Y^{-1} = (a_{ij})$. Then, for $1 \le j < i \le n$, we have $|a_{ij}| \le f_{i-j}$, where f_{i-j} is the $(i-j)$-th Fibonacci number.*

Proof. Let $N = (n_{ij})$ be as in Lemma 2.2. Let $j = 1, 2, \dots, n-1$. We prove by induction on $t = 1, 2, \dots, n-j$ that $|a_{j+t,j}| \le f_t$ for all $Y \in K_n$. By Lemma 2.2, we have $|a_{j+1,j}| = n_{j+1,j}$, where $n_{j+1,j}$ can be 0 or 1. Thus, $|a_{j+1,j}| \le 1 = f_1$ for all $Y \in K_n$. Now, assume that for each $t = 1, 2, \dots, k-1$ we have $|a_{j+t,j}| \le f_t$ for all $Y \in K_n$. We prove that $|a_{j+k,j}| \le f_k$ for all $Y \in K_n$. By Lemma 2.2, we have $a_{j+k,j} = -\sum_{i=j}^{j+k-1} n_{j+k,i}a_{ij}$ for all $Y \in K_n$.

Case 1. Assume $n_{j+k,j+k-1} = 0$. Then $|a_{j+k,j}| = \left|\sum_{i=j}^{j+k-2} n_{j+k,i}a_{ij}\right|$. Also, by Lemma 2.2, $a_{j+k-1,j} = -\sum_{i=j}^{j+k-2} n_{j+k-1,i}a_{ij}$. Since both of $n_{j+k,i}$ and $n_{j+k-1,i}$ for each $i = j, j+1, \dots, j+k-2$ can arbitrarily be 0 or 1, it is clear that $a_{j+k,j}$ and $a_{j+k-1,j}$ go through the same values, as Y goes through the set K_n. Therefore, by the induction hypothesis, we obtain $|a_{j+k,j}| \le f_{k-1} \le f_k$ for all $Y \in K_n$.

Case 2. Assume $n_{j+k,j+k-1} = 1$.

Subcase i. Assume $n_{j+k-1,j+k-2} = 0$. By Lemma 2.2, we have

$$|a_{j+k,j}| \le \left|\sum_{i=j}^{j+k-2} n_{j+k,i}a_{ij}\right| + |a_{j+k-1,j}|.$$

Also, by Lemma 2.2, it is clear that $a_{j+k-1,j} = -\sum_{i=j}^{j+k-2} n_{j+k-1,i}a_{ij}$. Since both of $n_{j+k,i}$ and $n_{j+k-1,i}$ for each $i = j, j+1, \dots, j+k-2$ can arbitrarily be 0 or 1, it is clear that $-\sum_{i=j}^{j+k-2} n_{j+k,i}a_{ij}$ and $a_{j+k-1,j}$ go through the same values, as Y goes through the set K_n. Thus, by the induction hypothesis, we obtain $\left|\sum_{i=j}^{j+k-2} n_{j+k,i}a_{ij}\right| \le f_{k-1}$. Beside this, by our assumption in Subcase i, $|a_{j+k-1,j}| = \left|\sum_{i=j}^{j+k-3} n_{j+k-1,i}a_{ij}\right|$. Since both of $n_{j+k-1,i}$ and $n_{j+k-2,i}$ for each $i = j, j+1, \dots, j+k-3$ can arbitrarily be 0 or 1, it is obvious that $\sum_{i=j}^{j+k-3} n_{j+k-1,i}a_{ij}$ and $a_{j+k-2,j}$ go through the same values, as Y goes through the set K_n. By the induction hypothesis, we obtain $|a_{j+k-1,j}| \le f_{k-2}$ for all $Y \in K_n$. Thus, $|a_{j+k,j}| \le f_{k-1} + f_{k-2} = f_k$ for all $Y \in K_n$.

Subcase ii. Assume $n_{j+k-1,j+k-2} = 1$. By Lemma 2.2, we have

$$|a_{j+k,j}| \le \left|\sum_{i=j}^{j+k-3} n_{j+k,i}a_{ij}\right| + \left|\sum_{i=j+k-2}^{j+k-1} n_{j+k,i}a_{ij}\right|.$$

Since $\sum_{i=j}^{j+k-3} n_{j+k,i}a_{ij}$ and $a_{j+k-2,j}$ go through the same values, as Y goes through the set K_n, by the induction hypothesis, we have $\left|\sum_{i=j}^{j+k-3} n_{j+k,i}a_{ij}\right| \le f_{k-2}$ for all $Y \in K_n$. In addition to this, since $n_{j+k-1,j+k-2} = 1$ we obtain

$$\sum_{i=j+k-2}^{j+k-1} n_{j+k,i}a_{ij} = (n_{j+k,j+k-2} - 1)a_{j+k-2,j} - \sum_{i=j}^{j+k-3} n_{j+k-1,i}a_{ij}.$$

Here each $(1 - n_{j+k,j+k-2}), n_{j+k-1,j}, \dots, n_{j+k-1,j+k-3}$ can arbitrarily be 0 or 1. Thus, $\sum_{i=j+k-2}^{j+k-1} n_{j+k,i}a_{ij}$ and $a_{j+k-1,j}$ go through the same values, as Y goes through the set K_n and hence, by the induction hypothesis, $\left|\sum_{i=j+k-2}^{j+k-1} n_{j+k,i}a_{ij}\right| \leq f_{k-1}$ for all $Y \in K_n$. Therefore, we obtain $|a_{j+k,j}| \leq f_{k-2} + f_{k-1} = f_k$ for all $Y \in K_n$.

The principle of induction completes the proof. $\square$

Let $A = (a_{ij})$, $B = (b_{ij}) \in M_n(\mathbb{R})$, that is, the set of all $n \times n$ real matrices. We write $A \geq 0$ if all $a_{ij} \geq 0$. Also, we write $A \geq B$ if $A - B \geq 0$. In addition to this, we define $|A| = (|a_{ij}|)$, that is, $|A|$ is the element-wise absolute value of A. The largest eigenvalue of A in modulus is denoted by $\rho(A)$ and called the spectral radius of A. Now we fix the notation for the rest of the paper.

Theorem 2.2. *Let $Y_0 = (y_{ij}^0) \in K_n$ be as in (1.2) and let $Z_0 := Y_0Y_0^T$. For all $Z \in L_n$, we have $|Z^{-1}| \leq |Z_0^{-1}|$.*

Proof. Firstly we obtain the inverse of Y_0. Let $N_0 = Y_0 - I$ and $N_0 = (m_{ij})$. Then

$$m_{ij} = \begin{cases} 0 & \text{if } \ i \leq j, \\ \frac{1-(-1)^{i+j}}{2} & \text{otherwise.} \end{cases}$$

We claim that the inverse of Y_0 is the $n \times n$ matrix (c_{ij}), where

$$c_{ij} = \begin{cases} 0 & \text{if } \ i < j, \\ 1 & \text{if } \ i = j, \\ (-1)^{i-j} f_{i-j} & \text{if } \ i > j. \end{cases}$$

Since $Y_0 \in K_n$ by Lemma 2.2, it is clear that $c_{ij} = 0$ if $i < j$ and $c_{ij} = 1$ if $i = j$. Also, by Lemma 2.2, we have

$$c_{ij} = -\sum_{k=j}^{i-1} m_{ik}c_{kj}$$

for $i > j$. Now we prove that $c_{ij} = (-1)^{i-j} f_{i-j}$ whenever $i > j$ by induction on $t = i - j$. For $t = 1$,

$$c_{j+1,j} = -m_{j+1,j} = -1 = -f_1.$$

Assume that $c_{j+t,j} = (-1)^t f_t$ for all $t = 1, 2, \dots, k - 1$. Recall that

$$c_{j+k,j} = -\sum_{s=j}^{j+k-1} m_{j+k,s}c_{sj}.$$

By the induction hypothesis, if k is even then we have

$$c_{j+k,j} = \sum_{s=1}^{k/2} f_{2s-1} = f_k$$

and if k is odd then

$$c_{j+k,j} = -1 - \sum_{s=1}^{(k-1)/2} f_{2s} = -f_k.$$

Here the last equalities follow from the well-known Fibonacci identities, see [22, Theorem 5.2 and Corollary 5.1]. Thus, $c_{j+k,j} = (-1)^k f_k$.

Secondly, we calculate the inverse of Z_0. Since $Z_0 = Y_0 Y_0^T$ and $Y_0^{-1} = (c_{ij})$, we have

$$(Z_0^{-1})_{ii} = \sum_{k=1}^{n} c_{ki}^2 = 1 + \sum_{k=i+1}^{n} f_{k-i}^2 \tag{2.2}$$

for all $i = 1, 2, \ldots, n$. Now let $1 \leq i < j \leq n$. Then

$$\begin{aligned}
(Z_0^{-1})_{ij} &= \sum_{t=1}^{n} c_{ti} c_{tj} \\
&= c_{ji} + \sum_{t=j+1}^{n} c_{ti} c_{tj} \\
&= (-1)^{j-i} f_{j-i} + \sum_{t=j+1}^{n} (-1)^{-i-j} f_{t-i} f_{t-j} \\
&= (-1)^{j-i} (f_{j-i} + \sum_{t=j+1}^{n} f_{t-i} f_{t-j}). \qquad (2.3)
\end{aligned}$$

Since Z_0^{-1} is symmetric, for all $1 \leq j < i \leq n$,

$$(Z_0^{-1})_{ij} = (-1)^{i-j} (f_{i-j} + \sum_{t=i+1}^{n} f_{t-i} f_{t-j}). \tag{2.4}$$

Now we prove the claim of the theorem. For each $Z \in L_n$, there exists a matrix Y in K_n such that $Z = YY^T$. Let $Y^{-1} = (a_{ij})$. Then, by Lemma 2.2 and Theorem 2.1, we have

$$\begin{aligned}
\left|(Z^{-1})_{ii}\right| &= \left|\sum_{k=1}^{n} a_{ki}^2\right| \\
&= \sum_{k=1}^{n} |a_{ki}|^2 \\
&= \sum_{k=i}^{n} |a_{ki}|^2 \\
&\leq 1 + \sum_{k=i+1}^{n} f_{k-i}^2 \\
&= \left|(Z_0^{-1})_{ii}\right|
\end{aligned}$$

for all $i = 1, 2, \dots, n$. Let $j > i$. By Lemma 2.2 and Theorem 2.1, we have

$$\begin{aligned}
\left|(Z^{-1})_{ij}\right| &= \left|\sum_{t=1}^{n} a_{ti}a_{tj}\right| \\
&\leq \sum_{t=1}^{n} |a_{ti}||a_{tj}| \\
&= |a_{ji}| + \sum_{t=j+1}^{n} |a_{ti}||a_{tj}| \\
&\leq f_{j-i} + \sum_{t=j+1}^{n} f_{t-i}f_{t-j} \\
&= \left|(Z_0^{-1})_{ij}\right|.
\end{aligned}$$

Finally, since Z^{-1} and Z_0^{-1} are symmetric, $|Z^{-1}| \leq |Z_0^{-1}|$ for all $Z \in L_n$. $\square$

3. Proof of Conjecture 1.1

The following lemma is crucial in the proof of Conjecture 1.1.

Lemma 3.1. *(See [18, Theorem 8.1.18].) Let $A, B \in M_n(\mathbb{R})$. If $|A| \leq B$, then $\rho(A) \leq \rho(|A|) \leq \rho(B)$.*

Proof of the conjecture. Let Z_0 be as in Theorem 2.2. First we prove that the matrices Z_0^{-1} and $|Z_0^{-1}|$ have the same characteristic polynomial. By the definition of the trace of a square matrix, it is clear that

$$\operatorname{tr}((Z_0^{-1})^k) = \sum_{i_1,\dots,i_k=1}^{n} (Z_0^{-1})_{i_1 i_2} \dots (Z_0^{-1})_{i_{k-1} i_k} (Z_0^{-1})_{i_k i_1}$$

for each $k = 1, 2, \ldots, n$. Also, from formulae (2.2)–(2.4) in the proof of Theorem 2.2, one can easily show that $sgn(Z_0^{-1})_{ij} = (-1)^{i-j}$ for all $1 \le i, j \le n$. Thus, we have

$$sgn((Z_0^{-1})_{i_1 i_2} \ldots (Z_0^{-1})_{i_{k-1} i_k} (Z_0^{-1})_{i_k i_1}) = 1$$

and hence $\mathrm{tr}(|Z_0^{-1}|^k) = \mathrm{tr}((Z_0^{-1})^k)$. By Newton's identities [20], we obtain that Z_0^{-1} and $|Z_0^{-1}|$ have the same characteristic polynomial. Thus, $\rho(|Z_0^{-1}|) = \rho(Z_0^{-1})$. From Theorem 2.2 and Lemma 3.1, now we obtain

$$\rho(Z^{-1}) \le \rho(|Z^{-1}|) \le \rho(|Z_0^{-1}|) = \rho(Z_0^{-1}),$$

for all Z in L_n. Since all Z in L_n are positive definite, the smallest eigenvalue of Z_0 is less than or equal to the smallest eigenvalue of Z for all Z in L_n. $\square$

Finally, in our investigation [2], we cannot find any matrix Y other than Y_0 in K_n such that c_n is equal to the smallest eigenvalue of YY^T for each $n = 2, 3, \ldots, 9$. After this observation we can present the following conjecture.

Conjecture 3.1. *Let $n \ge 2$. There is a unique matrix Y in K_n such that c_n is equal to the smallest eigenvalue of YY^T. In other words, if the smallest eigenvalue of YY^T is equal to c_n then $Y = Y_0$, where Y_0 is defined by (1.2).*

4. Lower bounds for c_n

In the literature there are not so many results on estimating the value of c_n. Recently, Mattila [24] has presented a lower bound for c_n. Indeed, he proved that c_n is bounded below by $(\frac{6}{n^4+2n^3+2n^2+n})^{\frac{n-1}{2}}$. Then he showed that this lower bound can be replaced with $(\frac{48}{n^4+56n^2+48n})^{\frac{n-1}{2}}$ when n is even, and $(\frac{48}{n^4+50n^2+48n-51})^{\frac{n-1}{2}}$ when n is odd. Recently, beside Mattila's results, Altınışık and Büyükköse [4] have obtained a lower bound for the smallest eigenvalue t_n of the $n \times n$ matrix $E_n^T E_n$, where the ij-entry of E_n is 1 if $j|i$ and 0 otherwise, i.e., $t_n \ge \left(n \sum_{k=1}^{n} \mu^2(k)\right)^{-1}$. Indeed, this bound can be used instead of lower bounds including c_n for the smallest eigenvalues of GCD and related matrices defined on $S = \{1, 2, \ldots, n\}$ in the literature, see [14,16,19,24,27]. After above studies on estimating the value of c_n, we naturally raise the following problem.

Problem 4.1. Can one improve the lower bounds mentioned above for c_n?

In the review process, Professor Jorma K. Merikoski proposed a solution to Problem 4.1, which is presented in Appendix A.

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Appendix A. A solution to Problem 4.1 (by Jorma K. Merikoski)

Recall that the matrix $Z_0^{-1} = (\zeta_{ij})$ satisfies

$$\zeta_{ii} = 1 + \sum_{k=i+1}^{n} f_{k-i}^2, \qquad |\zeta_{ij}| = f_{|j-i|} + \sum_{k=j+1}^{n} f_{k-i} f_{k-j}, \ i \neq j.$$

Also recall that the spectral radius of a nonnegative square matrix is less than or equal to the maximal row sum, and equality holds if and only if the matrix is irreducible and all row sums are equal (e.g., [9, Chapter 2, Theorem (2.35)]). Now

$$\begin{aligned} c_n^{-1} = \rho(Z_0^{-1}) = \rho(|Z_0^{-1}|) < \max_i \sum_{j=1}^{n} |\zeta_{ij}| &= \sum_{j=1}^{n} |\zeta_{1j}| \\ &= |\zeta_{11}| + \sum_{j=2}^{n} |\zeta_{1j}| \\ &= 1 + \sum_{k=2}^{n} f_{k-1}^2 + \sum_{j=2}^{n} \Big(f_{j-1} + \sum_{k=j+1}^{n} f_{k-1} f_{k-j} \Big) \\ &= 1 + \sum_{k=1}^{n-1} f_k + \sum_{j=1}^{n} \sum_{k=j+1}^{n} f_{k-1} f_{k-j} \\ &= 1 + S_1 + S_2. \end{aligned}$$

It is well-known [12, Eq. (22)] that

$$\sum_{k=1}^{m} f_k = f_{m+2} - 1; \tag{A.1}$$

so

$$S_1 = f_{n+1} - 1.$$

More work must be done with S_2. First we have

$$\begin{aligned} S_2 = (f_1^2 + f_2^2 + \cdots + f_{n-1}^2) + (f_2 f_1 + f_3 f_2 + \cdots + f_{n-1} f_{n-2}) + \cdots \\ + (f_j f_1 + f_{j+1} f_2 + \cdots + f_{n-1} f_{n-j}) + \cdots + (f_{n-2} f_1 + f_{n-1} f_2) + f_{n-1} f_1 \end{aligned}$$

$$
\begin{aligned}
&= f_1(f_1 + \cdots + f_{n-1}) + f_2(f_2 + \cdots + f_{n-1}) + \cdots + f_j(f_j + \cdots + f_{n-1}) \\
&\quad + \cdots + f_{n-2}(f_{n-2} + f_{n-1}) + f_{n-1}^2 \\
&= \sum_{j=1}^{n-1} f_j \sum_{k=j}^{n-1} f_k.
\end{aligned}
$$

By (A.1),

$$
\sum_{k=j}^{n-1} f_k = \sum_{k=1}^{n-1} f_k - \sum_{k=1}^{j-1} f_k = (f_{n+1} - 1) - (f_{j+1} - 1) = f_{n+1} - f_{j+1},
$$

which implies

$$
S_2 = \sum_{j=1}^{n-1} f_j(f_{n+1} - f_{j+1}) = f_{n+1} \sum_{j=1}^{n-1} f_j - \sum_{j=1}^{n-1} f_j f_{j+1} = T_1 - T_2.
$$

It remains to simplify T_1 and T_2. By (A.1),

$$
T_1 = f_{n+1}(f_{n+1} - 1).
$$

It is well-known [29, Sequence A064831] that

$$
\begin{aligned}
&\textstyle\sum_{j=1}^{m} f_j f_{j+1} = f_{m+1}^2 - 1 \text{ if } n \text{ is even}, \\
&\textstyle\sum_{j=1}^{m} f_j f_{j+1} = f_{m+1}^2 \text{ if } n \text{ is odd}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&T_2 = f_n^2 \text{ if } n \text{ is even}, \\
&T_2 = f_n^2 - 1 \text{ if } n \text{ is odd}.
\end{aligned}
$$

In other words,

$$
T_2 = f_n^2 - \eta_n,
$$

where

$$
\eta_n = \frac{1 - (-1)^n}{2}.
$$

Now

$$
\begin{aligned}
c_n^{-1} &< 1 + S_1 + T_1 - T_2 \\
&= 1 + (f_{n+1} - 1) + f_{n+1}(f_{n+1} - 1) - f_n^2 + \eta_n
\end{aligned}
$$

$$
\begin{aligned}
&= f_{n+1}^2 - f_n^2 + \eta_n \\
&= (f_{n+1} - f_n)(f_{n+1} + f_n) + \eta_n \\
&= f_{n-1} f_{n+2} + \eta_n .
\end{aligned}
$$

It is well-known [8, Eq. (2)] that

$$f_{m-r} f_{m+s} - f_m f_{m+s-r} = (-1)^{m-r-1} f_r f_s .$$

Hence

$$f_{n-1} f_{n+2} = f_n f_{n+1} + (-1)^n ,$$

and so

$$c_n^{-1} < f_n f_{n+1} + (-1)^n + \eta_n = f_n f_{n+1} + \theta_n ,$$

where

$$\theta_n = \frac{1 + (-1)^n}{2} .$$

Therefore

$$c_n > (f_n f_{n+1} + \theta_n)^{-1} = \gamma_n ,$$

which seems to be much better than Mattila's [25] bounds. For example, $c_6 = 0.0148$ (in three digits). The bound $\gamma_6 = 0.00952$ is rather good, while Mattila's better bound is 0.0000205.

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